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Polyhedra dominating finitely many different homotopy types

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Abstract

In 1968 K. Borsuk asked: Is it true that every finite polyhedron dominates only finitely many different shapes? In this question the notions of shape and shape domination can be replaced by the notions of homotopy type and homotopy domination.

We obtained earlier a negative answer to the Borsuk question and next results that the examples of such polyhedra are not rare. In particular, there exist polyhedra with nilpotent fundamental groups dominating infinitely many different homotopy types. On the other hand, we proved that every polyhedron with finite fundamental group dominates only finitely many different homotopy types. Here we obtain next positive results that the same is true for some classes of polyhedra with Abelian fundamental groups and for nilpotent polyhedra. Therefore we also get that every finitely generated, nilpotent torsion-free group has only finitely many r-images up to isomorphism.

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An introduction

By a polyhedron we mean, as usual, a finite one. Each polyhedron and CW-complex is assumed to be connected (for convenience).

In 1968 at the Topological Conference in Herceg-Novi K. Borsuk stated the following:

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Problem. Does every polyhedron dominate only finitely many different shapes?

See [2]; see also [1] for an equivalent version of the question, for *FANR*'s.
(The basic notions and results of shape theory the reader can find in [3,18,7].)

In the above problem the notions of shape and shape domination can be replaced by the notions of homotopy type and homotopy domination (respectively). Indeed, by the known results in shape theory (from [8,10], [7, Theorems 2.2.6, and 2.1.6]) we get that, for each polyhedron P, there is a 1–1 functorial correspondence between the shapes of compacta shape dominated by P and the homotopy types of CW-complexes (not necessarily finite) homotopy dominated by P (in both pointed and unpointed cases).

In the sequel we will concentrate on the pointed version of the question. By the results of [10] and [6, Theorem 5.1] we obtain that in pointed and unpointed cases the answer is the same.

Recall that each space homotopy dominated by a polyhedron has the homotopy type of a CW-complex, not necessarily finite (1949 J.H.C. Whitehead; see also [24]). Thus the Borsuk problem is equivalent to:

Problem. Does every polyhedron homotopy dominate only finitely many different homotopy types?

In this paper we consider dominations of a polyhedron in the category of CW-complexes and homotopy classes of cellular maps between them.

When the question was stated, it was known that every 1-dimensional polyhedron dominates only finitely many different shapes. It is a consequence of the result of S. Trybulec [23] that every movable curve has a plane shape and the theorem of K. Borsuk stating that two plane continua have the same shape if and only if their Betti numbers coincide [3, Theorem 7.1, p. 221].

By the result of S. Mather [17] (see also a simple paper of Holsztyński [12]), every polyhedron dominates only a countable number of different homotopy types (hence shapes). Directly in shape category the same was proven by Moron and Ruiz del Portal in [19].

In [16] we showed that generally an answer to the Borsuk question is negative: there exists a polyhedron (even of dimension 2), which homotopy dominates infinitely many polyhedra of different homotopy types.

Moreover, we proved that such examples are not rare: for every non-Abelian poly-$\mathbb{Z}$-group G and an integer $n \geqslant 3$ there exists a polyhedron P with $\pi_1(P) \cong G$ and $\dim P = n$ dominating infinitely many polyhedra of different homotopy types (see [13]). Thus, there exist polyhedra with nilpotent fundamental groups with this property.

On the other hand, in [15] we obtained, using the results of localization theory in the homotopy category of CW-complexes, that every simply-connected polyhedron dominates only finitely many different homotopy types. In [14] we proved, in an other way, that polyhedra with finite fundamental groups dominate only finitely many different homotopy types.

In this paper, extending the methods of [14], we prove that for some classes of polyhedra with Abelian fundamental group, the answer to the Borsuk question is positive.

We also show that every nilpotent polyhedron dominates only finitely many different homotopy types. Hence we get that every finitely generated, nilpotent torsion-free group has only finitely many r-images up to isomorphism.

Preliminaries

Definition 1. By a *normal series* of a group G we mean any descending sequence

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k = 1,$$

of its normal subgroups $G_i \lhd G$ (for $i = 1, \ldots, k$). The quotient groups G_{i-1}/G_i are the *factors* of the series.

Definition 2. We define the *lower central series* of a group G,

$$G = G_{(0)} \supseteq G_{(1)} \supseteq \cdots \supseteq G_{(i)} \supseteq G_{(i+1)} \supseteq \cdots$$

by setting $G_{(i+1)} = [G, G_{(i)}]$, for each $i \geqslant 0$. A group G is *nilpotent* if $G_{(i)} = 1$ for i sufficiently large.

Definition 3. A group G is *polycyclic* if G has a normal series for which each factor G_{i-1}/G_i is infinite cyclic or finite cyclic. The number of infinite cyclic terms occurring in a decomposition as above is an invariant of G (independent of the series) known as the *Hirsch number* of G. A group G is a *poly-$\mathbb{Z}$-group* if it has a normal series with factors $\mathbb{Z}$.

Remark. Recall that, for finitely generated groups we have the following, more or less known, facts:

(1) $0 \subseteq$ Abelian groups $\subseteq$ nilpotent groups $\subseteq$ polycyclic groups (for the last see, for example, [22, 15.4, p. 92]).
(2) Torsion-free nilpotent groups $\subseteq$ poly-$\mathbb{Z}$-groups (see, for instance, [21, Theorem 5.2.20]).
(3) Every poly-$\mathbb{Z}$-group is finitely presented (Hall, see [21, pp. 53, 54]).

In [13] we proved the following:

Theorem [13, Theorem 2]. *For every non-Abelian poly-$\mathbb{Z}$-group G and an integer $n \geqslant 3$, there exists a polyhedron P with $\pi_1(P) \cong G$, $\dim P = n$ dominating infinitely many different homotopy types of polyhedra.*

Corollary [13, Corollary 5]. *There exist polyhedra with nilpotent fundamental groups dominating infinitely many different homotopy types.*

Nilpotent polyhedra

In this section we show that every nilpotent polyhedron dominates only finitely many different homotopy types.

Definition 4. A CW-complex X is *nilpotent* if X is connected, $\pi_1(X)$ is nilpotent and, for every integer $i > 1$, $\pi_1(X)$ acts nilpotently on $\pi_i(X)$.

An action of a group G on an Abelian group M is *nilpotent* if there exists a finite series

$$0 \subseteq M_0 \subseteq \cdots \subseteq M_k = M,$$

such that G acts trivially on M_i/M_{i-1}, for $i = 1, 2, \dots, k$.

Definition 5. Recall that a CW-complex X is *simple* if $\pi_1(X)$ is Abelian and acts trivially on all higher homotopy groups $\pi_i(X)$ ($i \geqslant 2$).

Example 1.

(a) All the simply-connected finite CW-complexes, and all the simple finite CW-complexes (hence finite H-spaces) are nilpotent.
(b) All the Eilenberg–MacLane CW-complexes $K(G, 1)$, for finitely generated nilpotent torsion-free groups G, are nilpotent finite CW-complexes of dimension equal to the Hirsch number of G, up to homotopy type (see, for example, [4, Chapter 8, Theorem 7.1 and Chapter 6]).

Example 2. An example of a non-nilpotent finite CW-complex with an Abelian fundamental group is the projective plane $\mathbb{R}P^2$ (and all the projective spaces $\mathbb{R}P^n$, for even $n \in \mathbb{N}$).

Definition 6. A map $f : X \to Y$ between two CW-complexes X and Y is said to be a *homology equivalence* if it induces an isomorphism of the integral homology groups $H_i(f) : H_i(X; \mathbb{Z}) \to H_i(Y; \mathbb{Z})$, for all $i \in \mathbb{N}$ (see [25, pp. 181–182]).

Let $X \leqslant P$ denote that X is homotopy dominated by P.

Theorem 1. *Every nilpotent polyhedron dominates only finitely many different homotopy types.*

Proof. Theorem 1 of [14] states that for every polyhedron P, all the CW-complexes $X \leqslant P$ can be partitioned into finitely many classes such that for any two CW-complexes belonging to the same class, there is a homology equivalence between them.

One can see that, if P is a nilpotent polyhedron, then every CW-complex $X \leqslant P$ is also nilpotent. To prove the theorem, we use a case of theorem of Dror from [5] (see [9, Theorem, p. 259]) that, if a map $f : X \to Y$ of connected, nilpotent CW-complexes is a homology equivalence, then it is a homotopy equivalence. $\square$

Definition 7. A homomorphism $f : G \to H$ of groups is an r-homomorphism if there exists a homomorphism $g : H \to G$ such that $fg = \mathrm{id}_H$. Then H is an r-*image* of G.

As a corollary to Theorem 1, we obtain:

Corollary 1. *Let G be a finitely generated nilpotent torsion-free group. Then G has only finitely many r-images up to isomorphism.*

Proof. For every finitely generated nilpotent torsion-free group G, an Eilenberg–MacLane CW-complex $K(G,1)$ has the homotopy type of a finite, nilpotent, CW-complex of dimension equal to the Hirsch number of G (compare Example 1).

Every r-image of a finitely generated, torsion-free and nilpotent group has such specified properties (for the last see, for example, [21, 5.1.4]).

Let us associate with each retract H of the group G a finite CW-complex $K(H,1)$. Recall that for every homomorphism of groups $\varphi : \pi \to \rho$, there exists an unique (up to homotopy) pointed map $f : K(\pi,1) \to K(\rho,1)$ such that $\pi_1(f) = \varphi$ (see [25, Theorem 4.3, p. 225]). Thus, if H is a retract of G, then $K(H,1)$ is homotopy dominated by $K(G,1)$. We apply Theorem 1 and the proof is finished. $\square$

Let us note that an answer to the following question is unknown:

Question 1. Does there exist a finitely presented group which has infinitely many different r-images up to isomorphism?

Polyhedra with Abelian fundamental groups

Since there exist polyhedra with nilpotent fundamental groups homotopy dominating infinitely many different homotopy types, one may ask the following:

Question 2. Does every polyhedron P with the Abelian fundamental group $\pi_1(P)$ dominate only finitely many different homotopy types?

We now extend the methods used in the case of polyhedra with finite fundamental groups (see [14]) to some classes of polyhedra with Abelian fundamental groups.

Definition 8. Recall that an *idempotent* in a category $\mathcal{C}$ is a morphism k such that $k \circ k = k$ in $\mathcal{C}$.

Let X be a CW-complex. Writing $X \leqslant P$, we will always mean a triple (X, d_X, u_X) where $d_X : P \to X$ is a fixed domination of P over X with a fixed inverse map $u_X : X \to P$, i.e., $d_X u_X \simeq \mathrm{id}_X$. Then $k_X = u_X d_X : P \to P$ is an idempotent in the homotopy category of CW-complexes and homotopy classes of cellular maps between them.

In what follows, $\mathcal{M}_n(K)$ denotes the set of all matrices $n \times n$ over a field K.

In the proof of Theorem 2 we will use the following three lemmas (proofs of Lemmas 2 and 3, and more about Lemma 1 with Corollary 2 can be found in [14]).

The proof of the following lemma by M. Putcha is a consequence of the Hilbert Basis Theorem.

Lemma 1 ([20, Lemma 1.6], see also [14]). *Let $\mathcal{S}$ be an infinite set of idempotents in $\mathcal{M}_n(K)$ of* rank *r, where K is a given algebraically closed field. Then there exist two idempotents $E, F \in \mathcal{S}$ such that $E \neq F$ and* $\operatorname{rank}(EF) = \operatorname{rank}(FE) = r$.

Corollary 2 (Compare [14]). *Let K be an algebraically closed field. Then any given subset $\mathcal{S}$ of idempotents in $\mathcal{M}_n(K)$ of rank r can be partitioned into finitely many classes such that E and F belong to the same one if and only if there exist a finite set of idempotents $E_0 = E, E_1, \ldots, E_m = F$, where $E_j \in \mathcal{S}$, such that for each $1 \leqslant j \leqslant m$,*

$$\operatorname{rank}(E_{j-1}E_j) = \operatorname{rank}(E_j E_{j-1}) = r.$$

In the sequel $\mathbb{Z}/\mathbf{p}$ will denote the integers modulo a prime p.

Lemma 2 [14, Lemma 2]. *Let $f : X \to Y$, where X and Y are CW-complexes with finitely generated integral homology groups. Suppose that f induces an isomorphism $H_{i,\mathbb{Q}}(f) : H_i(X;\mathbb{Q}) \to H_i(Y;\mathbb{Q})$, for all $i \in \mathbb{N}$. Then f induces also isomorphisms $H_{i,p}(f) : H_i(X;\mathbb{Z}/\mathbf{p}) \to H_i(Y;\mathbb{Z}/\mathbf{p})$, for all $i \in \mathbb{N}$ and primes $p \in S$, where $S = \{p \in primes \mid p$ does not divide any of the torsion coefficients of the groups $H_i(X;\mathbb{Z})$, $H_i(Y;\mathbb{Z})$, for $i = 1, 2, \ldots\}$.*

Lemma 3 [14, Lemma 3]. *Let $f : X \to Y$, where X and Y are CW-complexes with finitely generated integral homology groups, induce, for all $i \in \mathbb{N}$, isomorphisms $H_{i,\mathbb{Q}}(f) : H_i(X;\mathbb{Q}) \to H_i(Y;\mathbb{Q})$ and $H_{i,p}(f) : H_i(X;\mathbb{Z}/\mathbf{p}) \to H_i(Y;\mathbb{Z}/\mathbf{p})$, for each prime p. Then f is a homology equivalence.*

In what follows, $\widetilde{X}$ will denote as usual, the universal covering space of X.

Recall that if $f : X \to Y$ is a cellular map of CW-complexes such that $f(x) = y$, for vertices $x \in X$, $y \in Y$, then for choosen $\tilde{x} \in p^{-1}(x)$ and $\tilde{y} \in q^{-1}(y)$ (where p, q denote the suitable projections, respectively), there exists a unique map $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ such that $q\tilde{f} = fp$ and $\tilde{f}(\tilde{x}) = \tilde{y}$. The map $\tilde{f}$ induces homomorphisms $\tilde{f}_i : H_i(\widetilde{X}) \to H_i(\widetilde{Y})$, for $i = 1, 2, \ldots$, and one can say that $\tilde{f}_i$ is induced by f (see [11, p. 107]). In the sequel it will be convenient to assume (without loss of generality) that every CW-complex X has just one vertex $x \in X$ and choose a base point $\tilde{x} \in p^{-1}(x)$ for $\widetilde{X}$. (In our notations the base points will be omitted).

Theorem 2. *Suppose that P is a polyhedron such that $\pi_1(P)$ is Abelian, and the groups $H_i(\widetilde{P};\mathbb{Z})$ are finitely generated, for $i \geqslant 2$. Then P dominates only finitely many different homotopy types.*

Proof. The proof is similar to that of [14, Theorem 2]. We will show the following:

Suppose that P is a polyhedron such that $\pi_1(P)$ is Abelian and the groups $H_i(\widetilde{P};\mathbb{Z})$ are finitely generated, for $i \geqslant 2$. All the CW-complexes homotopy dominated by P can be partitioned into finitely many classes, such that if X and Y belong to the same one, then there exists a finite sequence $\{X_j\}_{j=0}^m$, $m \geqslant 1$, where $X_0 = X$, $X_m = Y$, $X_j \leqslant P$, such that the map

$$g_{XY} = d_Y k_{X_{m-1}} k_{X_{m-2}} \cdots k_{X_2} k_{X_1} u_X : X \to Y$$

induces an isomorphism $\pi_1(g_{XY}) : \pi_1(X) \to \pi_1(Y)$, and the lifting of g_{XY} to the universal covers

$$\tilde{g}_{XY} = \tilde{d}_Y \tilde{k}_{X_{m-1}} \tilde{k}_{X_{m-2}} \cdots \tilde{k}_{X_2} \tilde{k}_{X_1} \tilde{u}_X : \widetilde{X} \to \widetilde{Y}$$

induces isomorphisms $H_i(\tilde{g}_{XY}) : H_i(\widetilde{X};\mathbb{Z}) \to H_i(\widetilde{Y};\mathbb{Z})$, for all $i \in \mathbb{N}$.

In the first step, take a set $T = \{p \in primes \mid p$ divides some of the torsion coefficients of the groups $H_1(P;\mathbb{Z}), H_2(P;\mathbb{Z}), \ldots, H_n(P;\mathbb{Z})$ or $H_2(\widetilde{P};\mathbb{Z}), H_3(\widetilde{P};\mathbb{Z}), \ldots, H_n(\widetilde{P};\mathbb{Z})\}$, where $n = \dim P$.

We can divide all the CW-complexes homotopy dominated by P into finitely many classes such that if X and Y belong to the same class, then, for each $p \in T$, the map $f_{XY} = d_Y u_X : X \to Y$ induces isomorphisms $H_{i,p}(f_{XY}) : H_i(X;\mathbb{Z}/\mathbf{p}) \to H_i(Y;\mathbb{Z}/\mathbf{p})$ and its lifting to the universal covers $\tilde{f}_{XY} = \tilde{d}_Y \tilde{u}_X : \widetilde{X} \to \widetilde{Y}$, isomorphisms $H_{i,p}(\tilde{f}_{XY}) : H_i(\widetilde{X};\mathbb{Z}/\mathbf{p}) \to H_i(\widetilde{Y};\mathbb{Z}/\mathbf{p})$, for all $i \in \mathbb{N}$.

To do this, observe that, for all $i \in \mathbb{N}$, $H_i(\widetilde{P};\mathbb{Z}/\mathbf{p})$ and $H_i(P;\mathbb{Z}/\mathbf{p})$ are finite. Let X and Y belong to the same class if and only if, for all the $p \in T$ and $i \in \mathbb{N}$, $\operatorname{im} H_{i,p}(u_X) = \operatorname{im} H_{i,p}(u_Y)$ in $H_i(P;\mathbb{Z}/\mathbf{p})$ and $\operatorname{im} H_{i,p}(\tilde{u}_X) = \operatorname{im} H_{i,p}(\tilde{u}_Y)$ in $H_i(\widetilde{P};\mathbb{Z}/\mathbf{p})$, where $H_{i,p}(u_X) : H_i(X;\mathbb{Z}/\mathbf{p}) \to H_i(P;\mathbb{Z}/\mathbf{p})$, $H_{i,p}(\tilde{u}_X) : H_i(\widetilde{X};\mathbb{Z}/\mathbf{p}) \to H_i(\widetilde{P};\mathbb{Z}/\mathbf{p})$ are the homomorphisms induced by u_X and its lifting to the universal covers $\tilde{u}_X$, respectively (compare the proof of Lemma 4 in [14]).

In the sequel it will be convenient to prove and use the following:

Let P be a polyhedron and the groups $H_i(\widetilde{P};\mathbf{Z})$ are finitely generated, for $i \geqslant 2$. Then any subclass $\mathcal{W}$ of all the CW-complexes homotopy dominated by P can be partitioned into finitely many classes, such that X and Y belong to the same one if and only if there exists a finite sequence $\{X_j\}_{j=0}^m$, $m \geqslant 1$, where $X_0 = X$, $X_m = Y$, $X_j \in \mathcal{W}$, such that the map

$$g_{XY} = d_Y k_{X_{m-1}} k_{X_{m-2}} \cdots k_{X_2} k_{X_1} u_X : X \to Y$$

induces isomorphisms $H_{i,\mathbb{Q}}(g_{XY}) : H_i(X;\mathbb{Q}) \to H_i(Y;\mathbb{Q})$, and the map

$$\tilde{g}_{XY} = \tilde{d}_Y \tilde{k}_{X_{m-1}} \tilde{k}_{X_{m-2}} \cdots \tilde{k}_{X_2} \tilde{k}_{X_1} \tilde{u}_X : \widetilde{X} \to \widetilde{Y}$$

induces isomorphisms $H_{i,\mathbf{Q}}(\tilde{g}_{XY}) : H_i(\widetilde{X};\mathbb{Q}) \to H_i(\widetilde{Y};\mathbb{Q})$, for all $i \in \mathbb{N}$.

For the proof of this statement, suppose that $X \in \mathcal{W}$. Observe that there is a 1–1 correspondence between idempotent homomorphisms of a vector space $H_i(P;\mathbb{Q})$ and idempotent matrices in $\mathcal{M}_{l_i}(\mathbb{Q})$, where $l_i = \beta_i(P)$. The same is true for homomorphisms of $H_i(\widetilde{P};\mathbb{Q})$ and idempotent matrices in $\mathcal{M}_{s_i}(\mathbb{Q})$, where $s_i = \beta_i(\widetilde{P})$.

Let $M_i(X) \in \mathcal{M}_{l_i}(\mathbb{Q})$ ($l_i = \beta_i(P)$), for $i = 1, 2, \ldots, n$, be a matrix of the homomorphism

$$H_{i,\mathbb{Q}}(k_X) : H_i(P;\mathbb{Q}) \to H_i(P;\mathbb{Q})$$

induced by the idempotent $k_X : P \to P$.

Similarly, let $N_i(X) \in \mathcal{N}_{s_i}(\mathbb{Q})$ ($s_i = \beta_i(\widetilde{P})$), for $i = 2, \ldots, n$, be a matrix of the homomorphism

$$H_{i,\mathbb{Q}}(\tilde{k}_X) : H_i(\widetilde{P}; \mathbb{Q}) \to H_i(\widetilde{P}; \mathbb{Q})$$

induced by the lifting of k_X to the universal covers $\tilde{k}_X : \widetilde{P} \to \widetilde{P}$.

Take a $\mathbb{Q}$-vector space $V = \bigoplus_{i=1}^n H_i(P; \mathbb{Q}) \oplus \bigoplus_{i=2}^n H_i(\widetilde{P}; \mathbb{Q})$. Let $M(X) \in \mathcal{M}_l(\mathbb{Q})$, where $l = \sum_{i=1}^n l_i + \sum_{i=1}^n s_i$, be a matrix of the homomorphism

$$k_{X_*} = \bigoplus_{i=1}^n H_{i,Q}(k_X) \oplus \bigoplus_{i=2}^n H_{i,Q}(\tilde{k}_X) : V \to V.$$

$M(X)$ is then a matrix composed by idempotent matrices $M_i(X)$ and $N_i(X)$ as follows:

$$M(X) = \begin{bmatrix} M_1(X) & & & & & & & & \\ & M_2(X) & & & & & \textit{zeroes} & & \\ & & \ddots & & & & & & \\ & & & M_n(X) & & & & & \\ & & & & N_2(X) & & & & \\ & & & & & N_3(X) & & & \\ & \textit{zeroes} & & & & & \ddots & & \\ & & & & & & & N_n(X) \end{bmatrix}.$$

Observe that then $M(X)$ is an idempotent matrix. We will apply Corollary 2 to the class $\mathcal{S}$ of all the matrices $M(X) \in \mathcal{M}_l(\mathbf{Q})$, for $X \in \mathcal{W}$. We may assume, without loss of generality, that the collections ($\operatorname{rank} M_1(X), \operatorname{rank} M_2(X), \ldots, \operatorname{rank} M_n(X), \operatorname{rank} N_2(X), \operatorname{rank} N_3(X), \ldots, \operatorname{rank} N_n(X)$) of ranks of the matrices $M_i(X)$ and $N_i(X)$ are the same.

By Corollary 2, we obtain a partition of all the matrices $M(X)$, and then all the CW-complexes $X \in \mathcal{W}$, into finitely many classes such that X and Y belong to the same one if and only if there exists a sequence $\{X_j\}_{j=0}^m$ ($m \geqslant 1$) of CW-complexes $X_j \in \mathcal{W}$, where $X_0 = X$, $X_m = Y$, such that $\operatorname{rank} M(X_{j-1})M(X_j) = \operatorname{rank} M(X_j)M(X_{j-1}) = \operatorname{rank} M(X_{j-1}) = \operatorname{rank} M(X_j)$, for $j = 1, 2, \ldots, m$.

Then $\operatorname{rank} M_i(X_{j-1})M_i(X_j) = \operatorname{rank} M_i(X_j)M_i(X_{j-1}) = \operatorname{rank} M_i(X_{j-1}) = \operatorname{rank} M_i(X_j)$ and $\operatorname{rank} N_i(X_{j-1})N_i(X_j) = \operatorname{rank} N_i(X_j)N_i(X_{j-1}) = \operatorname{rank} N_i(X_{j-1}) = \operatorname{rank} N_i(X_j)$, for all i and $j = 1, 2, \ldots, m$ (compare the proof of Lemma 1 in [14]).

Therefore that the map

$$g_{XY} = d_Y k_{X_{m-1}} k_{X_{m-2}} \cdots k_{X_2} k_{X_1} u_X : X \to Y$$

induces isomorphisms $H_{i,\mathbb{Q}}(g_{XY}) : H_i(X; \mathbb{Q}) \to H_i(Y; \mathbb{Q})$ for all $i \geqslant 1$, and the map

$$\tilde{g}_{XY} = \tilde{d}_Y \tilde{k}_{X_{m-1}} \tilde{k}_{X_{m-2}} \cdots \tilde{k}_{X_2} \tilde{k}_{X_1} \tilde{u}_X : \widetilde{X} \to \widetilde{Y},$$

isomorphisms $H_{i,\mathbb{Q}}(\tilde{g}_{XY}) : H_i(\widetilde{X}; \mathbb{Q}) \to H_i(\widetilde{Y}; \mathbb{Q})$, for all $i \geqslant 2$.

Observe that then, by Lemma 2, for each prime $p \notin T$ and integer $i \in \mathbb{N}$, $g_{XY} : X \to Y$ induces also an isomorphism $H_{i,p}(g_{XY}) : H_i(X; \mathbb{Z}/\mathbf{p}) \to H_i(Y; \mathbb{Z}/\mathbf{p})$ and $\tilde{g}_{XY} : \widetilde{X} \to \widetilde{Y}$ induces an isomorphism $H_{i,p}(\tilde{g}_{XY}) : H_i(\widetilde{X}; \mathbb{Z}/\mathbf{p}) \to H_i(\widetilde{Y}; \mathbb{Z}/\mathbf{p})$.

We may assume that the same is true for each prime $p \in T$, taking as $\mathcal{W}$ a class of CW-complexes dominated by P after the partition in the first step of the proof (compare the proof of Lemma 4 in [14]).

Applying Lemma 3, we conclude that all the CW-complexes dominated by P can be partitioned into finitely many classes such that X and Y belong to the same one if and only if there exists a sequence $\{X_j\}_{j=0}^m$ of CW-complexes $X_j \in \mathcal{W}$, where $X_0 = X$, $X_m = Y$, such that the map

$$g_{XY} = d_Y k_{X_{m-1}} k_{X_{m-2}} \cdots k_{X_2} k_{X_1} u_X : X \to Y$$

induces isomorphisms $H_i(g_{XY}) : H_i(X; \mathbb{Z}) \to H_i(Y; \mathbb{Z})$, for all $i \in \mathbb{N}$, and the map

$$\tilde{g}_{XY} = \tilde{d}_Y \tilde{k}_{X_{m-1}} \tilde{k}_{X_{m-2}} \cdots \tilde{k}_{X_2} \tilde{k}_{X_1} \tilde{u}_X : \widetilde{X} \to \widetilde{Y},$$

isomorphisms $H_{i,Z}(\tilde{g}_{XY}) : H_i(\widetilde{X}; \mathbb{Z}) \to H_i(\widetilde{Y}; \mathbf{Z})$, for all $i \in \mathbb{N}$.

Thus $g_{XY} : X \to Y$ induces an isomorphism $\pi_1(g_{XY}) : \pi_1(X) \to \pi_1(Y)$, while $\tilde{g}_{XY} : \widetilde{X} \to \widetilde{Y}$ induces isomorphisms $H_i(\tilde{g}_{XY}) : H_i(\widetilde{X}; \mathbb{Z}) \to H_i(\widetilde{Y}; \mathbb{Z})$, for all $i \in \mathbf{N}$. By the Whitehead Theorem (see, for example, [11, Theorem 3.1, p. 107]), $g_{XY} : X \to Y$ is a homotopy equivalence. This finishes the proof. □

Example 3. A simple example of a polyhedron P with Abelian $\pi_1(P)$ and infinitely generated $H_2(\widetilde{P}; \mathbb{Z})$ is $P = S^2 \vee S^1$.

Remark. The author expects that Theorem 2 can be generalized for polyhedra P with nilpotent fundamental group $\pi_1(P)$ and $H_i(\widetilde{P}; \mathbb{Z})$ finitely generated, for $i \geqslant 2$.

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